M.A./M.Sc. Mathematics, Entrance Exam 2020

1. Let $\{x_n\}$ and $\{y_n\}$ be sequences of real numbers such that $x_n \leq y_n$ for all $n \geq N$, where N is some positive integer. Consider the following statements:
   (a) $\liminf_{n\to\infty} x_n \leq \liminf_{n\to\infty} y_n$.
   (b) $\limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} y_n$.
   Which of the above statements is(are) correct?
    1. Neither (a) nor (b).
    2. Only (a).
    3. Only (b).
    4. Both (a) and (b).

2. Which of the sequences $\{a_n\}$ and $\{b_n\}$ of real numbers with n-th terms
$$a_n = \frac{(n^2+20n+35)\sin n^3}{n^2+n+1}, \quad b_n = 2\cos n - \sin n$$
   has(have) convergent subsequences?
    1. Neither $\{a_n\}$ nor $\{b_n\}$.
    2. Only $\{a_n\}$.
    3. Only $\{b_n\}$.
    4. Both $\{a_n\}$ and $\{b_n\}$.

3. Consider the following series:
   (a) $\sum_{n=1}^{\infty} \frac{x^n}{n!}, \quad x \in \mathbb{R}$
   (b) $\sum_{n=1}^{\infty} \frac{1}{n+\sin n}$
   (c) $\sum_{n=1}^{\infty} \frac{1}{2^n\sqrt{n}}$
   (d) $\sum_{n=1}^{\infty} \sin n$
   Which of the above series is(are) convergent?
    1. All of (a), (b), (c) and (d).
    2. Only (a), (c) and (d).
    3. Only (a) and (c).
    4. Only (c).

4. The union of infinitely many closed subsets of the real line is
    1. uncountable.
    2. finite.
    3. always closed.
    4. need not be closed.

5. Consider the series $\sum_{n=1}^{\infty} a_n$ where $a_n = \left(2+\sin\frac{n\pi}{2}\right)r^n$, $r>0$. What are the values of $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n}$ and $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n}$?
    1. $r/2$ and $2r$.
    2. $r/3$ and r.
    3. $2r/3$ and $3r/2$.
    4. 0 and 1.

6. Consider the following series:

(a) $\sum_{n=1}^{\infty} 3^{-n} \sin(3^n x)$ on $\mathbb{R}$

(b) $\sum_{n=1}^{\infty} 2^{-n} x^n$ on $(-2, 2)$

(c) $\sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$ on $\mathbb{R}$

Which of the above series converge uniformly on the indicated domain?

1. Only (a) and (b).
2. Only (b) and (c).
3. Only (a) and (c).
4. All of (a), (b) and (c).

7. Let $\{f_n\}$ be a sequence of continuous functions on $[a, b]$ converging uniformly to the function f. Consider the following statements:

(a) f is bounded on $[a, b]$.

(b) $\lim_{n\to\infty} \int_a^b f_n(t)dt = \int_a^b f(t)dt$.

(c) If each f_n is differentiable, then the sequence $\{f_n'\}$ converges uniformly to f' on $[a, b]$, f' is the derivative of f.

Which of the following statements is(are) correct?

1. Only (a) and (b)
2. Only (a) and (c)
3. Only (c).
4. Only (b).

8. Let $G(x)$ be a real valued function defined by $G(x) = \int_{x^2}^{4x^2} \cos\sqrt{t}dt$. If G' is the derivative of G, then

1. $G'(\pi/2) = -4\pi$.
2. $G'(\pi/2) = -4\pi - 1$.
3. $G'(\pi/2) = -\pi$.
4. $G'(\pi/2) = 0$.

9. Let

$$f(x) = \begin{cases} (4 - x^2)^{5/2}, & |x| < 2 \\ 0, & |x| \geq 2 \end{cases}.$$

Consider the following statements:

(a) f is not continuous on $\mathbb{R}$

(b) f is continuous on $\mathbb{R}$ but not differentiable at $x = 2, -2$

(c) f is differentiable on $\mathbb{R}$ but f' is not continuous on $\mathbb{R}$

(d) f is differentiable on $\mathbb{R}$ and f' is continuous on $\mathbb{R}$

Which of the above statements is(are) correct?

1. Only (a) and (d).
2. Only (b) and (c).
3. Only (c).
4. Only (d).

10. The zero of the function $f(x) = -2x^3 + 5x - 5$ defined on $\mathbb{R}$ lie in the interval

1. $(-1, 1)$.
2. $[3, 4]$.
3. $[-2, -1]$.
4. $[-5, -3]$.

11. The Wronskian of $\cos x$, $\sin x$ and e^{-x} at $x = 0$ is
    1. 1.
    2. 2.
    3. -1.
    4. -2.

12. The solution of the initial value problem $y' = 1 + y^2,\ y(0) = 1$, is
    1. $y = \operatorname{cosec}(x + \pi/4)$.
    2. $y = \tan\left(x + \pi/4\right)$.
    3. $y = \sec\left(x + \pi/4\right)$.
    4. $y = \cot\left(x + \pi/4\right)$.

13. How many solution(s) does the initial value problem $y' - \dfrac{2}{x}y = 0,\ y(0) = 0$ have?
    1. No solution.
    2. Unique solution.
    3. Two solutions.
    4. Infinitely many solutions.

14. The general solution of the equation $y'' + y = \operatorname{cosec} x$, $(0 < x < \pi)$, is (c_1, c_2 are arbitrary constants)
    1. $c_1 \cos x + c_2 \sin x - x \cos x + \sin x \ln(\sin x)$.
    2. $c_1 \cos x + c_2 \sin x + x \cos x + \sin x \ln(\sin x)$.
    3. $c_1 \cos x + c_2 \sin x - x \sin x + \cos x \ln(\sin x)$.
    4. $c_1 \cos x + c_2 \sin x + x \sin x + \cos x \ln(\sin x)$.

15. The particular integral of the differential equation $y'' + y = x^3$ is
    1. $x^2 + 6x$.
    2. $x^2 - 6x$.
    3. $x^3 + 6x$.
    4. $x^3 - 6x$.

16. The complete integral of the partial differential equation $p^2z^2 + q^2 = 1$, where $p = \dfrac{\partial z}{\partial x}, q = \dfrac{\partial z}{\partial y}$ is (a, b are arbitrary constants)
    1. $z + a^2 \ln\left(\dfrac{z + \sqrt{z^2 + a^2}}{a}\right) = 0$.
    2. $a^2 z + by + x^2 = 0$.
    3. $z\sqrt{z^2 + a^2} + a^2 \ln\left(\dfrac{z + \sqrt{z^2 + a^2}}{a}\right) = 2x + 2ay + 2b$.
    4. $z^2 + y^2 = x^2 + 2x + 2ay + 2b$.

17. The complete integral of the partial differential equation $z = px + qy - \sin(pq)$, where $p = \dfrac{\partial z}{\partial x}, q = \dfrac{\partial z}{\partial y}$, is
    1. $z = ax + by + \sin(ab)$.
    2. $z = ax + by - \sin(ab)$.
    3. $z = ax + y + \sin b$.
    4. $z = x + by - \sin a$.

18. The partial differential equation $yu_{xx} + 2xyu_{xy} + xu_{yy} = u_x + u_y$ is
    1. hyperbolic in $\{(x, y) \mid 0 < xy < 1\}$.
    2. hyperbolic in $\{(x, y) \mid xy > 1\}$.

3. elliptic in $\{(x,y) \mid xy > 1\}$.
4. elliptic in $\{(x,y) \mid xy < 0\}$.

19. The general solution of the equation $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2} = x - y$ is (ϕ_1, ϕ_2 are arbitrary functions)

    1. $\dfrac{1}{4}x(x-y)^2 + \phi_1(x^2+y) + \phi_2(x-y)$.
    2. $\dfrac{1}{4}x(x-y)^2 + \phi_1(x+y) + \phi_2(x-y)$.
    3. $\phi_1(x+y) + \phi_2(x^2-y)$.
    4. $\phi_1(x^2+y) + \phi_2(x^2-y) - \dfrac{1}{4}x(x+y)$.

20. The solution of the partial differential equation $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ with $u(0,t) = u(2,t) = 0$, $u(x,0) = \sin^3 \dfrac{\pi x}{2}$ and $u_t(x,0) = 0$ is

    1. $\dfrac{3}{4} \sin \dfrac{\pi x}{2} \sin \dfrac{\pi c t}{2}$.
    2. $\dfrac{3}{4} \sin \dfrac{\pi x}{2} \cos \dfrac{\pi c t}{2} - \dfrac{1}{4} \sin \dfrac{3\pi x}{2} \cos \dfrac{3\pi c t}{2}$.
    3. $\dfrac{3}{4} \cos \dfrac{\pi x}{2} \sin \dfrac{\pi c t}{2} - \dfrac{1}{4} \sin \dfrac{3\pi x}{2} \sin \dfrac{3\pi c t}{2}$.
    4. $\dfrac{3}{4} \sin \dfrac{\pi x}{2} \cos \dfrac{\pi c t}{2} - \dfrac{1}{4} \cos \dfrac{3\pi x}{2}$.

21. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} (x^2+y^2)\ln(x^2+y^2), & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}.$$

    Then

    1. f_{xy} and f_{yx} are continuous at $(0,0)$, and $f_{xy}(0,0) = f_{yx}(0,0)$.
    2. f_{xy} and f_{yx} are discontinuous at $(0,0)$, but $f_{xy}(0,0) = f_{yx}(0,0)$.
    3. f_{xy} and f_{yx} are continuous at $(0,0)$, but $f_{xy}(0,0) \neq f_{yx}(0,0)$.
    4. f_{xy} and f_{yx} are discontinuous at $(0,0)$, and $f_{xy}(0,0) \neq f_{yx}(0,0)$.

22. The directional derivative of $f(x,y,z) = xy^2 + yz^2 + zx^2$ defined on $\mathbb{R}^3$ along the tangent to the curve $x = t, y = t^2, z = t^3$ at the point $(1,1,1)$ is

    1. $-\dfrac{18}{\sqrt{14}}$.
    2. $\dfrac{13}{\sqrt{14}}$.
    3. $-\dfrac{13}{\sqrt{14}}$.
    4. $\dfrac{18}{\sqrt{14}}$.

23. The unique polynomial of degree 2 passing through $(1,1), (3,27)$ and $(4,64)$ obtained by Lagrange interpolation is

    1. $8x^2 - 17x + 12$.
    2. $8x^2 - 19x - 12$.
    3. $8x^2 + 14x - 12$.
    4. $8x^2 - 19x + 12$.

24. The approximate value of $\displaystyle\int_0^3 \frac{dx}{(1+x)^2}$ by Simpson's 1/3-rd rule, using the least number of equal subintervals, is

1. 0.8512.
2. 0.8125.
3. 0.7625.
4. 0.6702.

25. Consider the differential equation $\frac{dy}{dx} = y - x$, $y(0) = 2$. The absolute value of the difference in the solutions obtained by Euler method and Runge–Kutta second order method at $y(0.1)$ using step size 0.1 is
    1. 2.205
    2. 2.252
    3. 0.005
    4. 0.055

26. The approximate value of $(17)^{1/3}$ obtained after two iterations of Newton–Raphson method starting with initial approximation $x_0 = 2$ is
    1. 2.7566
    2. 2.5826
    3. 2.6713
    4. 2.4566

27. For an infinite discrete metric space (X, d), which of the following statements is correct?
    1. X is compact.
    2. for every $A \subseteq X$, $A^o \cup \overline{A} = X$, where $\overline{A}$ and A^o denote respectively the closure and interior of A in X.
    3. X is connected.
    4. X is not totally bounded.

28. Consider the metric space (l_2, d) of square summable sequences in $\mathbb{R}$ with the Euclidean metric. Let $Y = \{e_1, e_2, \ldots\} \subseteq l_2$, where e_i is the sequence of 1 at the i-th place and 0 elsewhere. Then
    1. Y is not compact and has no limit point.
    2. Y is compact and each e_i is a limit point of Y.
    3. Y is not compact and has a limit point.
    4. Y is compact and has no limit point.

29. Let $C[0, 1]$ be the set of real valued continuous functions on $[0, 1]$ with the sup-metric. Let $A = \{f \in C[0, 1] \mid f(0) = 0\}$ and $B = \{f \in C[0, 1] \mid f(0) > 0\}$ be subspaces of $C[0, 1]$. Then
    1. both A and B are complete.
    2. A is complete but B is incomplete.
    3. A is incomplete but B is complete.
    4. neither A nor B is complete.

30. Let $(\mathbb{R}, d)$ and $(\mathbb{R}, u)$ be metric spaces with the discrete metric d and the usual metric u respectively. Let $f : (\mathbb{R}, d) \longrightarrow (\mathbb{R}, u)$ and $g : (\mathbb{R}, u) \longrightarrow (\mathbb{R}, d)$ be functions given by

$$f(x) = g(x) = \begin{cases} 0, & x \leq 0 \\ x + 1, & x > 0 \end{cases}.$$

    Then
    1. both f and g are continuous.
    2. neither f nor g is continuous.
    3. f is continuous but g is not.
    4. g is continuous but f is not.

31. Let $Y_1 = \{(x, y) \in \mathbb{R}^2 \mid y = \sin\frac{1}{x},\ 0 < x \leq \pi\}$ and $Y_2 = \{(0, y) \in \mathbb{R}^2 \mid y \in [-2, 2]\}$ be subspaces of the metric space $(\mathbb{R}^2, d)$, d being the Euclidean metric. For any $A \subseteq \mathbb{R}^2$, $\overline{A}$ denotes the closure of A in $\mathbb{R}^2$. Which of the following statements is correct?

1. $\overline{Y_1} \cup Y_2$ is connected.
2. $Y_1 \cup \overline{Y_2}$ is connected.
3. $\overline{Y_1} \cap Y_2$ is disconnected.
4. $\overline{Y_1 \cap Y_2}$ is a non-empty bounded subset of $\mathbb{R}^2$.

32. Let $R[0, 1]$ be the set of all real valued Riemann integrable functions on $[0, 1]$ and let $f : [0, 1] \longrightarrow \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ \frac{1}{n}, & \text{if } \frac{1}{n+1} < x \leq \frac{1}{n} \text{ for } n \in \mathbb{N} \end{cases}.$$

Which of the following statements is correct?

1. f is monotonically decreasing on $[0, 1]$ but $f \notin R[0, 1]$.
2. f is monotonically decreasing on $[0, 1]$ and $f \in R[0, 1]$.
3. f is discontinuous at infinitely many points in $[0, 1]$ but $f \notin R[0, 1]$.
4. f is discontinuous at infinitely many points in $[0, 1]$ and $f \in R[0, 1]$.

33. The improper integral $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2 + 1}$

1. converges to π.
2. converges to $\pi/2$.
3. converges to 0.
4. diverges.

34. Consider the functions $f(x) = \dfrac{x^2 - 1}{x - 1}$ and $g(x) = \dfrac{|x^2 - 1|}{x - 1}$, $x \neq 1$. Then

1. both f and g have removable discontinuity at $x = 1$.
2. both f and g have jump discontinuity at $x = 1$.
3. f has a removable discontinuity at $x = 1$, while g has a jump discontinuity at $x = 1$.
4. f has a jump discontinuity at $x = 1$ while g has a removable discontinuity at $x = 1$.

35. What is the length of the interval on which the function $f(x) = x^3 - 6x^2 - 15x + 8$ is decreasing?

1. 8
2. 6
3. 4
4. 2

36. Let $f : [a, b] \longrightarrow \mathbb{R}$ be a monotonic function. Consider the following statements:
(a) The function f obeys the maximum principle.
(b) The function f is Riemann integrable on $[a, b]$.
Which of the above statements is(are) correct?

1. Only (a).
2. Only (b).
3. Both (a) and (b).
4. Neither (a) nor (b).

37. Consider the following:
(a) $\langle (a, b), (c, d) \rangle = ac - bd,\ (a, b), (c, d) \in \mathbb{R}^2$.
(b) $\langle f(x), g(x) \rangle = \displaystyle\int_0^1 f'(x)g(x)dx$, where $f(x), g(x)$ are polynomials over $\mathbb{R}$.
Which of the above is(are) an inner product?

1. Neither (a) nor (b).
2. Both (a) and (b).
3. Only (a).
4. Only (b).

38. Let $T = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$. Then $-T^3 + 4T^2 + 5T - 2I$ is equal to
    1. $10T + 4I$.
    2. $10T - 4I$.
    3. $-10T + 4I$.
    4. $-10T - 4I$.

39. Let V be an infinite dimensional vector space over a field F. Consider the following statements:
    (a) Any one-one linear transformation from V to itself is onto.
    (b) Any onto linear transformation from V to itself must is one-one.
    Which of the above statements is(are) correct?
    1. Both (a) and (b).
    2. Only (a).
    3. Only (b).
    4. Neither (a) nor (b).

40. Let $P_n(\mathbb{R})$ be the set of all polynomials over $\mathbb{R}$ of degree atmost n. Let $T : P_n(\mathbb{R}) \longrightarrow P_{n+1}(\mathbb{R})$ be given by $T(f(x)) = xf(x)$. Then
    1. T is one-one and onto linear transformation.
    2. T is an onto function but neither a linear transformation nor one-one.
    3. T is not onto but a one-one linear transformation.
    4. T is one-one but neither a linear transformation nor onto.

41. Let $(\mathbb{Z}, *)$ be a group, where $a * b = a + b - 2$ and $\mathbb{Z}$ is the set of integers. The inverse of a is
    1. $a - 6$.
    2. $a - 4$.
    3. $4 - a$.
    4. $6 - a$.

42. Let G be a group of even order. Suppose that exactly half of G consists of elements of order 2 and the rest forms a subgroup H of G. Which of the following statements is incorrect?
    1. H is a normal subgroup of G.
    2. order of H is even.
    3. H is abelian.
    4. $[G : H] = 2$.

43. Let G and K be finite groups such that $|G| = 21$ and $|K| = 49$. Suppose G does not have a normal subgroup of order 3. Let L be the set of all group homomorphism from G to K. Then the number of elements in L is
    1. 1.
    2. 3.
    3. 5.
    4. 7.

44. Let G be a finite group and $a \in G$ has exactly two conjugates. Suppose that $C(a) = \{x^{-1}ax \mid x \in G\}$ and $N(a) = \{x \in G \mid ax = xa\}$. Which of the following statements is incorrect?
    1. The number of elements in $C(a)$ is a prime number.
    2. G is a simple group.

3. $N(a) \neq G$.
4. $N(a)$ is a normal subgroup of G.

45. Let G be a finite group of order 385. Let H, K and L be p-Sylow subgroups of G for $p = 5, 7$, and 11, respectively. Which of the following statements is incorrect?

1. K is a normal subgroup of G.
2. L is a normal subgroup of G.
3. HK is a non-abelian subgroup of G.
4. $G = HKL$.

46. The remainder when 2020^{2020} is divided by 12 is

1. 0.
2. 2.
3. 4.
4. 8.

47. The smallest integer $a > 2$ such that $2|a,\ 3|(a+1),\ 4|(a+2),\ 5|(a+3)$ and $6|(a+4)$ is

1. 14.
2. 56.
3. 122.
4. 62.

48. Let $R = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}$ be a ring and $f : R \longrightarrow \mathbb{Z}$ be given by $\phi\left(\begin{pmatrix} a & b \\ b & a \end{pmatrix}\right) = a - b$. Which of the following statements is incorrect?

1. ϕ is a ring homomorphism.
2. $\ker \phi$ is a prime ideal but not maximal.
3. $\ker \phi$ is a maximal ideal.
4. ϕ is surjective.

49. Consider the following statements:
(a) A polynomial is irreducible over a field F if it has no zeros in F.
(b) Let $f(x) \in \mathbb{Z}[x]$. If $f(x)$ is reducible over $\mathbb{Q}$, then it is reducible over $\mathbb{Z}$.
(c) For any prime p, the polynomial $x^{p-1} + x^{p-2} + \cdots + x + 1$ is irreducible over $\mathbb{Q}$.
Which of the above statements is(are) correct?

1. Only (a) and (b).
2. Only (a) and (c).
3. Only (b) and (c).
4. All of (a), (b) and (c).

50. Which of the following is a Euclidean domain?

1. $\mathbb{Q}[x]/\langle x^3 - 2\rangle$.
2. $\mathbb{Z}[x]$.
3. $\mathbb{Q}[x, y]$.
4. None of the above.